On the Estimation of the Mixing Matrix for Underdetermined Blind Source Separation in an Arbitrary Number of Dimensions*

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Abstract. Blind Source Separation consists of estimating n sources from the measurements provided by m sensors. In this paper we deal with the underdetermined case, $m < n$, where the solution can be implemented in two stages: first estimate the mixing matrix from the measurements and then estimate the best solution to the underdetermined linear problem. Instead of being restricted to the conventional two-measurements scenario, in this paper we propose a technique that is able to deal with this underdetermined linear problem at an arbitrary number of dimensions. The key points of our procedure are: to parametrize the mixing matrix in spherical coordinates, to estimate the projections of the maxima of the multidimensional PDF that describes the mixing angles through the marginals, and to reconstruct the maxima in the multidimensional space from the projections. The results presented compare the proposed approach with estimation using multidimensional ESPRIT.

1 Introduction

The blind source separation (BSS) problem consists of estimating n sources from the measurements provided by m sensors. In the noise-free linear model, the measurements are related to the sources through an unknown linear combination

$$\mathbf{A}\mathbf{s} = \mathbf{x}, \tag{1}$$

where $\mathbf{s} \in R^n$ is the source random vector, $\mathbf{x} \in R^m$ is the measurement random vector, and $\mathbf{A} \in R^{m \times n}$ is the unknown mixing matrix. Depending on the relation between m and n, we are faced with three different scenarios. The square $(m = n)$ and the strictly overdetermined $(m > n)$ cases have been extensively studied in the literature [1, 2], and all we need to separate the sources is to estimate the

* This work has been partially supported by Spanish Ministry of Science and Technology under project TIC2001-0751-C04-03.

C.G. Puntonet and A. Prieto (Eds.): ICA 2004, LNCS 3195, pp. 185–192, 2004.

mixing matrix $\mathbf{A}$, since the inverse solves the square problem, and the pseudo-inverse provides the solution with minimum-norm error in the overdetermined case [3].

The last scenario, in which we are interested in this paper, arises when the number of sensors is smaller than the number of sources ($m < n$). In this underdetermined case, the solution process can be divided in two stages: first estimate the mixing matrix from the measurements and then estimate the sources that "best" solve the underdetermined linear problem [4, 5]. This procedure relies on the premise that the sources are sparse or that a suitable linear transformation is applied to convert the non-sparse sources into a sparse representation [6]. To parametrically model sources with different degrees of sparsity, the following model for the source densities is used

$$p_{S_j}(s_j) = p_j\,\delta(s_j) + (1 - p_j) f_{S_j}(s_j), \quad j = 1, \ldots, n, \tag{2}$$

where s_j is the j-th source, p_j is the sparsity factor for s_j, and $f_{S_j}(s_j)$ is the PDF when the source j—that is assumed to be zero-mean—is active. The performance of this two-stage procedure strongly depends on the sparsity of the sources, both for the estimation of the mixing matrix and for [7] the estimation of the sources [8]: the higher the sparsity factor the better the estimation of mixing matrix and the recovery of the sources.

Most of the results on underdetermined BSS [6, 8] consider the case with two sensors ($m = 2$), in which the mixing matrix can be obtained, from a geometrical point of view [9], by finding the maxima of a unidimensional probability density function (PDF). However, the direct extension of this method to scenarios with more than two sensors requires finding the maxima of a multidimensional PDF [10], that, in addition to be computationally more complex, requires a number of samples that depends exponentially on the number of dimensions.

In this paper, we extend our previous work on underdetermined BSS [4] to deal with an arbitrary number of sensors (more than one) and an arbitrary number of sources. The organization of the paper is as follows: In Section 2, we present the problem of estimating the mixing matrix as the problem of finding the maxima of an $(m-1)$-dimensional PDF. In Section 3, we introduce the projection procedure that reduces the peak estimation problem from a multidimensional PDF to $m-1$ decoupled unidimensional PDFs, and show how to elucidate the spurious combinations of peaks from those that are true maxima of the $(m-1)$-dimensional PDF. In Section 4, we validate the proposed method with a series of Montecarlo simulations. In Section 5 we present the conclusions of this work.

2 Estimation of the Mixing Matrix

Equation (1) can be interpreted from a geometrical point of view as the projection of the source vectors $\mathbf{s}$ from R^n into the vector space R^m of the measurement vectors $\mathbf{x}$. If we denote by $\mathbf{a}_j$ the j-th column of the mixing matrix, so that $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n]$, (1) can be rewritten as

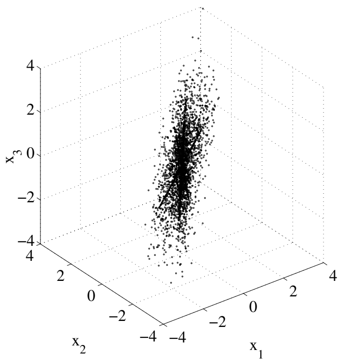

Fig. 1. Scatter plot of measurements for a scenario with three sensors ($m = 3$) and four sources ($n = 4$) of sparsity factor: 0.5.

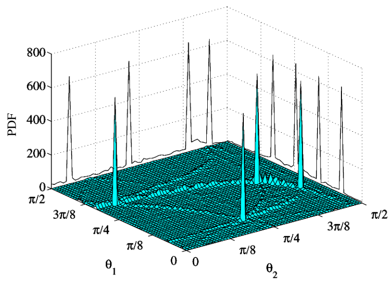

Fig. 2. Histogram of angles for the measurements of Figure 1. The $(m-1)$-unidimensional projections onto the plane of angle θ_i, $i = 1, \ldots, m-1$ are also shown.

$$\mathbf{x} = \sum_{j=1}^{n} s_j \mathbf{a}_j, \tag{3}$$

that explicitly shows that the measurement vector is a linear combination of the columns of the mixing matrix. According to this interpretation, if at a given time only the j-th source is non-zero, the measurement vector will be collinear with $\mathbf{a}_j$. When more than one source is active at the same time, the measurement will be a linear combination of the corresponding columns of the mixing matrix. In Figure 1 we show a scatter plot for a scenario with four sources and three sensors that is simulated for sources with sparsity factors of 0.5. For higher sparsity factors, the measurements are more concentrated along the directions of the columns of the mixing matrix [4].

The first step in our recovery procedure is to convert all the points of the m-dimensional vector space of the measurements and the columns of the mixing matrix from a Cartesian representation to a spherical coordinate system, where every point $\mathbf{x}$ of Cartesian coordinates $(x_1, \ldots, x_m)$ is represented by its modulus r and by $m-1$ angles θ_i. According to this definition, the angles can be determined from the rectangular coordinates as

$$\theta_i = \arctan \frac{x_{i+1}}{\sqrt{\sum_{l=1}^{i} x_l^2}}, \quad i = 1, \ldots, m-1. \tag{4}$$

If we apply (4) to the measurements of Figure 1, and represent an histogram taking as independent variables the $m-1$ angles, we obtain the results shown in Figure 2.

3 Dimension Reduction by Projection

Up to this point, we have reduced the problem of estimating the mixing matrix **A** to the problem of estimating the n peaks of an $(m-1)$-dimensional PDF, since those peaks define the spherical angles that parametrize the n columns of the mixing matrix. It is well known that the problem of estimating the peaks of a multidimensional PDF requires much more data samples as the dimensionality of the problem grows [11]. However, the idiosyncrasy of the underdetermined BSS problem will help us to circumvent this problem. The sparsity of the sources, which is a prerequisite for the proposed underdetermined BSS procedures to work, will be determinant to the ability of estimating a multidimensional PDF by means of unidimensional estimations. In Figure 2 it can be observed that the $(m-1)$-dimensional PDF is composed of a set of n peaks that, even for an sparsity factor of 0.5, are quite narrow. In Figure 3, a top view of the $(m-1)$-dimensional PDF is shown. The black spots correspond to the locations of the maxima from Figure 2.

Since we are interested in determining only the position of the peaks, and not the complete shape of the PDF, all the information we are looking for can be extracted from the $m-1$ projections onto the unidimensional vector spaces corresponding to conserving only one spherical coordinate and making zero all the other angles. These projections are shown in Figure 2 for the case of three sensors and four sources, which we are using as an example. They can be considered as the set of $m-1$ unidimensional PDFs of the $m-1$ spherical angles that are shown as projections in Figure 2.

To each of these $m-1$ unidimensional PDFs of the angles that parametrize the measurements, a method has to be applied to find up to n maxima, whose locations correspond to the estimates $\hat{\theta}_{ij}$, $i = 1, \ldots, m-1$, $j = 1, \ldots, n$. A number of methods could be applied, from the simpler one of calculating the histogram and finding the maxima, to the use of nonparametric estimation by means of Parzen windowing [7], or to the use of spectral estimation techniques suitable for the estimation of sinusoids in noise [12].

Once the estimations of the individual spherical angles are obtained, it is necesary to reconstruct the position of the maxima of the multidimensional PDF from the unidimensional projections. The problem arises from the loss of information inherent to the projection process, and can be visualized by reconsidering Figure 3. We are interested on the $(m-1)$-dimensional position of the maxima indicated by the black spots, but all we have access to from the unidimensional estimations is the projections of these spots onto each of the coordinate axes. From these projections, all the combinations of angles could be constructed, as it is shown with dotted lines in Figure 3, and a method has to be implemented that allows to distinguish the correct combinations from the spurious solutions.

Fortunately, there exist an easy way for the correct combinations to stand out: all that we need to do is to define a small area around each combination of angles, that constitutes a tentative solution, and count how many measurements fall into that area. The correct combinations will have a high number of occurrences, but a point falling into the region associated to a spurious combination will be an improbable event. Since the number of combinations of angles is n^{m-1}, the procedure to elucidate which are the correct combinations of angles is to construct an $(m-1)$-dimensional count array $\mathbf{C}$ of length n in each of the dimensions and find the maxima for each intersection of the $m-1$ dimensions. In our example of four sources and three sensors, the $(m-1)$-dimensional array is a 4×4 matrix. In equation (5) the calculated matrix for a simulation with sparsity factor 0.5 is shown. The matrix is shown upside-down to facilitate comparison with Figure 3. The higher the sparsity factor, the more concentrated the measurements along the columns of the mixing matrix. As an example, for a sparsity factor of 0.9, almost all the measurements fall into the regions associated with the correct combinations.

$$\mathbf{C}(0.5) = \begin{pmatrix} 669 & 0 & 1 & 2 \\ 2 & 3 & 705 & 1 \\ 0 & 1 & 3 & 632 \\ 1 & 674 & 3 & 0 \end{pmatrix}. \tag{5}$$

Since the method of estimating the peaks on the multidimensional space is based on the information obtained by projecting, a potential problem could appear when more than one peak is projected along any direction into the same point. In this situation, we would not detect the limit of up to n peaks in each coordinate, but a smaller number of peaks in some angles. However, this is not really a problem, because with the help of the count vector $\mathbf{C}$ we would detect the situation (there would be high count numbers for multiple combinations of the same angle, instead of a single maximum per row and column of $\mathbf{C}$) and we could estimate the position of all the peaks.

4 Numerical Results

To characterize the performance of our method, Montecarlo simulations have been performed to estimate the mixing matrix from scenarios with different numbers of sources and sensors. In all the cases, the source realizations have been generated according to the model in (2), using as $f_{S_j}(s_j)$, $j = 1, \ldots, m$, Gaussian densities with zero mean and unit variance. The simulations have been performed as follows: for each scenario, twenty thousand samples of sources with sparsity factors from 0.05 to 0.95 have been produced. For each scenario and sparsity factor, four hundred mixing matrices have been randomly generated, the spherical angles have been estimated from the unidimensional projected PDFs, and the criterion to select the correct combination of angles has been applied. The different scenarios considered are those associated with a number of sensors ranging from two to five, and a number of sources ranging from one to ten. As the figure of merit we have selected the number of errors in the estimation of the angles (defining a tolerance on the basis of the bin length used on the histograms). We define the mean error rate as the mean number of errors for all the mixing matrices divided by the total number of angles to estimate.

Figure 4 shows the results from scenarios with five sensors ($m = 5$) and a number of sources from six to twelve ($6 \leq n \leq 12$) for all the sparsity factors

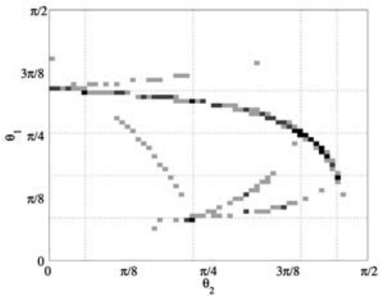

Fig. 3. Top view of the $(m-1)$-dimensional PDF corresponding to the spherical angles of the measurements. The black spots correspond to the locations of the maxima from Figure 2.

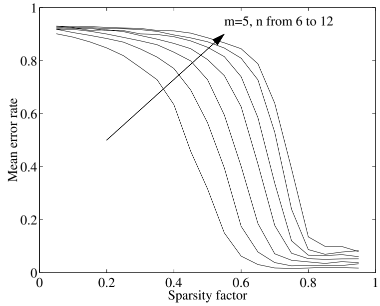

Fig. 4. Mean error rate for scenarios with a fixed number of five sensors and a number of sources ranging from six to twelve, as a function of the sparsity factor of the sources.

considered. It can be observed that the number of errors grows with the number of sources (more peaks have to be estimated from the same data, and the mean distance between peaks decreases), and diminish with the sparsity factor (the measurements tend to be more concentrated along the columns of the mixing matrix, reducing the spreading that confuses the estimation).

Figure 5 shows the results from scenarios with seven sources ($n = 7$) and a number of sensors ranging from two to six ($2 \leq m \leq 6$) for all the sparsity factors considered. It can be observed that the number of errors diminish as the number of available measurements increases.

Figure 6 shows the mean squared error (MSE) for the estimation of the angles of the mixing matrix for an scenario with four sources and two sensors obtained with the proposed reconstruction by projection method. In the same figure, the results obtained by applying two-dimensional ESPRIT to the direct estimation of the angles from the bidimensional PDF of Figure 2 are also shown. It is remarkable that the estimation from the projections, that is much easier and faster than the bidimensional ESPRIT, provides even better results.

5 Conclusions

In this paper we have presented a procedure to estimate the mixing matrix for underdetermined BSS problems in an arbitrary number of dimensions. The approach is based on parametrizing both the measurements and the columns of the mixing matrix in spherical coordinates and on estimating the peaks of the multidimensional PDF associated with the angles of the measurements. Since the estimation of multidimensional PDFs is a complex problem, we propose to project onto as many unidimensional PDFs as the number of spherical angles (the number of sensors minus one). Once the individual angles are estimated from the projections, the location of the peaks on the original multidimensional

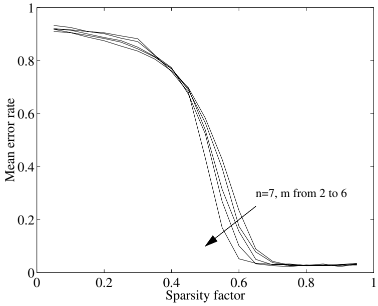

Fig. 5. Mean error rate for scenarios with a fixed number of seven sources and a number of sensors ranging from two to six, as a function of the sparsity factor of the sources.

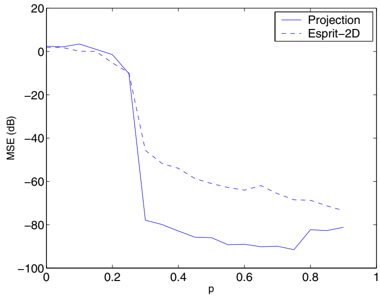

Fig. 6. MSE of the estimated angles $\hat{\theta}_{ij}$, $j = 1, \ldots, 4$ as a function of the sparsity factor (p) of the sources using both Esprit-2D (solid-line) and $m - 1$ projections (dashed-line) for an scenario with two sensors and four sources.

measurement space can be reconstructed. Since there exist different multidimensional PDFs compatible with the given projections, we propose a method to distinguish the spurious combinations of angles and to elucidate the correct combinations. We would like to point out that the procedure presented in this paper is not exclusive for underdetermined cases, since nothing prevents us from using this method in scenarios with less or equal sources than sensors. The reason why we focus on the underdetermined case is twofold: on the one hand, there exist other excellent approaches for the overdetermined and squared scenarios; on the other hand, the performance of our method increases with the sparsity factor of the sources, that is a prerequisite only for the underdetermined scenario. The Montecarlo simulations have shown that our method provides excellent results for an arbitrary number of sources and sensors provided that the sparsity factor is high enough (around 0.75). The intuitive result that the performance improves with the number of measurements and the sparsity factor, and degrades with the number of sources has also been corroborated.

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