The Impact of Pseudo-Measurements on State Estimator Accuracy

Kevin A. Clements

PSA Consulting

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The static state of an power system network is

\[ x = \begin{pmatrix} \theta \\ \phi \\ v \\ \delta \end{pmatrix} \in \mathbb{R}^n \]

where \( \theta \) is a vector of bus voltage angles, \( \phi \) is a vector of transformer phase shifts, \( v \) is a vector of bus voltage magnitudes and \( \delta \) is a vector of transformer off-nominal voltage ratios.
Network parameters and topology are assumed to be perfectly known in the state estimation problem formulation.

The available real-time measurements typically include:

- real and reactive line flows
- real and reactive bus injections
- bus voltage magnitudes
- line flow current magnitudes
- PMU phasor measurements
When the set of available real-time measurements is sufficient to allow a calculation of the system state vector, the system is said to be observable.

Otherwise the system is unobservable.

When the system is unobservable, there are observable islands as well as unobservable regions within the network.
The unobservable regions of the network can be estimated by using pseudo-measurements to augment the available real-time measurements.

Pseudo-measurements are typically calculated using short-term load forecasts or historical data.

Pseudo-measurements are much less accurate than the real-time measurements.
The unobservable regions of the network can be estimated by using pseudo-measurements to augment the available real-time measurements.

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Pseudo-measurements are much less accurate than the real-time measurements.
A state estimate is typically used to estimate line flows throughout the network.

These data, in turn, are used to assess system security.

The state estimate is used for real-time contingency analysis.

In this paper, which is tutorial in nature, we discuss the effect of pseudo-measurement errors on the accuracy of state, line flow and bus injection estimates.
The measurements, $z \in \mathbb{R}^m$, are modeled by the nonlinear measurement equation

$$z = h(x) + \epsilon$$

where $\epsilon$ is a vector of measurement errors.

Measurement error $\epsilon_i$ is modeled as a zero mean random variable whose standard deviation is $\sigma_i$. 
The power system state estimation problem is usually formulated as a nonlinear weighted least-squares (WLS) problem in which we seek to compute $x$ to minimize the objective function:

$$ f(x) = \frac{1}{2} [z - h(x)]^T R^{-1} [z - h(x)] $$

where $R$ is the $m \times m$ diagonal matrix

$$ R = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m^2 \end{pmatrix} $$
Newton’s method is widely used to solve systems of nonlinear equations.

When applied to the nonlinear least-squares problem it leads to the following iterative solution:

\[ x^{k+1} = x^k + B_k^{-1} H_k r_k \]

where

\[ r_k = z - h(x^k) \]

\[ H_k = H(x^k) = \nabla h(x^k) \]

and the Hessian matrix \( B_k \) is given by

\[ B_k = H_k^T R^{-1} H_k + \sum_{i=1}^{m} r_i^2 R^{-1} \nabla^2 r_i \]

Often, the second term in the above expression for \( B_k \) is ignored leading to the Gauss-Newton method.
In power system state estimation, the nonlinear least-squares problem is typically solved using the Gauss-Newton method which leads to the following iterative solution:

\[ x^{k+1} = x^k + G_k^{-1} H_k r_k \]

where

\[ G_k = H_k^T R^{-1} H_k \]

The Gauss-Newton method ignores the second order information contained in \( r_i^2 R^{-1} \nabla^2 r_i \).

The term \( r_i^2 R^{-1} \nabla^2 r_i \) will be small provided \( r_i^2 / \sigma_i^2 \) is small.
When a network is observable, $G_k$ is invertible which, in turn requires that the measurement Jacobian matrix, $H_k$ have full rank.

Network observability is determined by the type and location of the measurements in the network as well as the network topology.

Observability analysis can be done by direct evaluation of the rank of $G_k$, or by topological analysis of the measured network.

A critical measurement is one whose loss decreases the rank of $G_k$. 


The result of observability analysis is to identify islands of observability as well as unobservable branches.

A by-product of observability analysis is the identification of critical measurements.

As an example, consider the measured network shown on the following slide.
Network Observability and Critical Measurements

Example: An Eight Bus Measured Network

- Line Flow Measurement
- Bus Injection Measurement
Six line flow measurements and one injection measurement are shown on the previous slide.

There are also two bus voltage magnitude measurements, one at bus 1 and one at bus 6, that are not shown on the one line diagram.

Observability analysis of this network reveals that there are two observable islands and four unobservable branches.

The observable islands and unobservable branches are shown on the following slide.
Network Observability and Critical Measurements

Example: Eight Bus Measured Network

Figure: Observable Islands for the Eight Bus Example
In this example, at least one additional measurement, a pseudo-measurement, involving an unobservable branch power flow is required in order to attain observability.

This pseudo-measurement could be a bus injection measurement at either bus 2, or 3, or 4, or 5.
In this example, the measurements are partitioned into three sets:

\[
\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_C \end{pmatrix} = \begin{pmatrix} h_1(x_1) + \epsilon_1 \\ h_2(x_2) + \epsilon_2 \\ h_C(x_1, x_2, x_3) + \epsilon_C \end{pmatrix}
\]

The critical measurement \(z_C\) is the bus injection at bus 8 which is incident to unobservable branches.
In the general case, when a network is unobservable, there will be \( k \) observability islands and the measurement structure is given by

\[
    z = \begin{pmatrix}
        z_1 \\
        z_2 \\
        \vdots \\
        z_k \\
        z_C
    \end{pmatrix}
    = \begin{pmatrix}
        h_1(x_1) + \epsilon_1 \\
        h_2(x_2) + \epsilon_2 \\
        \vdots \\
        h_k(x_k) + \epsilon_k \\
        h_C(x_1, x_2, \ldots, x_k) + \epsilon_C
    \end{pmatrix}
\]

The measurement set \( z_C \) consists of measurements that are incident to unobservable branches.
The Branch Angle Transformation

- When multiple observable islands occur, it is necessary to choose a reference bus in each island when the state vector consists of bus voltage angles and bus voltage magnitudes.
- In order to avoid this complication, we can instead choose a set of $N-1$ branch angles to replace the bus voltage angles as state variables.
- The branch angles are related to the bus voltage angles by
  \[ \theta_B = A^T \theta \]
  where $A$ is the network node-to-branch incidence matrix.
Let $T$ be a set of $N - 1$ branches that form a spanning tree of the network, then we can partition $\theta_B$ into

$$\theta_B = \begin{pmatrix} \theta_T \\ \theta_L \end{pmatrix}$$

$\theta_T$ is sufficient to define all branch angles since the sum of branch angles around a fundamental loop must be 0. Therefore

$$\theta_L = -B_T \theta_T$$
The Branch Angle Transformation

- Replacing $\theta$ by $\theta_T$ in the state vector results in a bordered block diagonal structure for the measurement Jacobian matrix:

\[
H = \begin{pmatrix}
H_{11} & 0 & \cdots & 0 & 0 \\
0 & H_{22} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & H_{kk} & 0 \\
H_{C1} & H_{C2} & \cdots & H_{Ck} & H_{CC}
\end{pmatrix}
\]

where

\[
H_{ii} \in \mathbb{R}^{m_i \times n_i} \quad \text{and} \quad H_{CC} \in \mathbb{R}^{m_c \times n_c}
\]

- The diagonal blocks $H_{11}, \ldots, H_{kk}$ have full column rank provided that each observable island contains at least one voltage magnitude measurement.
The rank of $H_{CC}$ is $m_c$, the number of critical measurements incident to unobservable branches.

In order to attain observability, it is necessary to add $n_c - m_c$ suitable pseudo-measurements.

When the measurement set is augmented by critical pseudo-measurements in this manner, $H_{CC} \in \mathbb{R}^{n_c \times n_c}$ is an invertible matrix.
Error Analysis

- The WLS objective function can be written as

\[
f(x) = \frac{1}{2} \sum_{i=1}^{k} r_i(x_i)^T R_{ii}^{-1} r_i(x_i) + \frac{1}{2} r_C(x)^T R_C^{-1} r_C(x)
\]

- It is well known that the measurement residuals for critical measurements are zero. Consequently, the WLS objective function reduces to

\[
f(x) = \frac{1}{2} \sum_{i=1}^{k} r_i(x_i)^T R_{ii}^{-1} r_i(x_i)
\]

- The state estimates for the observable islands satisfy the first-order optimality conditions:

\[
H_{ii} R_{ii}^{-1} [z_i - h_i(x_i)] = 0
\]

- The state estimate for the unobservable part of the network satisfies

\[
h_C(x_C) = z_C
\]
Using the linearizing approximations

\[ h_i (\hat{x}_i) \approx h_i (x_i) + H_{ii} (\hat{x}_i) (\hat{x}_i - x_i) \]
\[ h_C (\hat{x}) \approx h_C (x) + H_C (\hat{x}) (\hat{x} - x) \]

we see that

\[ \tilde{x}_i = -G_{ii} (\hat{x}_i)^{-1} H_{ii}^T (\hat{x}_i) R_i^{-1} \epsilon_i \quad i = 1, \ldots, k \]

and

\[ \tilde{x}_C = -H_{CC}^{-1} \left( \epsilon_C + \sum_{i=1}^{k} H_{Ci} \tilde{x}_i \right) \]

where \( \tilde{x}_i = x_i - \hat{x}_i \) and \( \tilde{x}_C = x_C - \hat{x}_C \).
Shown below is the eight bus network with one pseudo-measurement added at bus 2.
In this example, neither observable island state estimate is affected by the pseudo-measurement.

The unobservable branches, of course, must use the pseudo-measurement data to calculate line flows and thus their accuracy is low compared with that of the observable islands.
Shown below is the eight bus network with pseudo-measurements added at buses 2 and 3.
As seen from the above error analysis, if only critical pseudo-measurements are used, the pseudo-measurement errors do not affect the accuracy of the state estimates for the observable islands.

If additional pseudo-measurements are used then pseudo-measurement errors will affect the accuracy of the state estimates in some or all of the observable islands.

Furthermore, if only critical pseudo-measurements are used, all pseudo-measurement residuals will be zero.

Large pseudo-measurement residuals may cause the Gauss-Newton approximation to the Hessian matrix to be poor and thereby contribute to slow convergence or non-convergence.

This effect can be ameliorated by assigning large $\sigma_i$’s to the pseudo-measurements.