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Asymptotic SNR-performance of some image combination techniques for phased-array MRI

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Abstract

Phased-array magnetic resonance imaging technology is currently flourishing with the promise of obtaining a profitable trade-off between image quality and image acquisition speed. The image quality is generally measured in terms of the signal-to-noise ratio (SNR), which is often calculated using samples taken from the reconstructed image. In this paper, we derive analytical expressions for the asymptotic SNR in the final image for three different phased-array image combination methods, namely: (1) sum-of-squares, (2) singular value decomposition, and (3) normalized coil averaging. The SNR expressions are expressed in terms of the statistics of the noise in the measurements, as well as the coil sensitivity coefficients. Our results can facilitate a better understanding for the phased-array image combination problem, as well as provide a tool for the optimal design of coils.

Keywords: Phased-array MRI; Multiple coils; SNR analysis

1. Introduction

Magnetic resonance imaging (MRI) as a noninvasive diagnostic tool has proven to be particularly valuable for the examination of the soft tissues in the body (such as the brain), and it has become an instrumental tool for the diagnosis of stroke and other significant diseases. MRI is also useful for pinpointing the focus of diseases such as epilepsy. Conventionally, MRI has been primarily used for imaging of static scenarios, but recently it has also been used with great success for imaging of time-varying processes. An important application of dynamic MRI is imaging of the heart and monitoring of cardiac diseases [8]; yet another example is functional MRI (fMRI), where the time variation of certain chemical compositions in the brain is investigated when a patient is subjected to an external stimulus. In these applications, the image acquisition speed is a critical, limiting factor.

Phased-array MRI involves a strong magnet and a number of radio antennas (coils) [1,2,5,7,9,10,12]. The main benefit of using a phased array is that by appropriately combining the signals from the different coils, the signal-to-noise ratio (SNR) can be improved, which gives an increase in image quality. Furthermore, since SNR can be traded for image

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acquisition speed, and increase in SNR leads to increased cost-effectiveness, reduced problems with motion artifacts, and decreased discomfort (such as long breath-holding durations) for patients.

In MRI, the image quality is typically evaluated using the SNR [3]. It is commonly accepted that gains in SNR above 20 dB are clinically not significant for *static imaging*, due to the limitations of the human visual perception. However, from the perspective of developing fast MR imaging technology, any improvement in SNR can be translated into a significant gain in imaging speed.

A difficulty associated with the evaluation of image reconstruction algorithm performance is that in general, it is often difficult to derive exact analytical expressions for the SNR associated with a given reconstruction algorithm, such as the popularly used sum-of-squares (SoS) method [2,10]. Recently, we have proposed two alternative image reconstruction algorithms [13], called the *singular value decomposition*(SVD) method, and the *normalized coil average* (NCA) method, which are based on the principle of maximum ratio combining—a technique approach borrowed from optimal array signal processing [11]. These reconstruction algorithms can be understood in an optimal second-order statistical framework (recall that mean-square-error (MSE) or SNR are both second-order statistical measures).

In this paper, we derive asymptotic expressions for the SNR in the reconstructed image when these three (SoS, SVD, and NCA) techniques are employed. The derived expressions are accurate for low noise coil measurement situations (i.e., when the SNR is high); nevertheless, they still can provide an insight into the low-SNR behavior of these algorithms. The presented analytical SNR expressions can be useful to design better phased array configurations, building strategies for the optimal usage of acquired coil measurements to determine the desired trade-off between speed and quality, and to establish an understanding for how model parameters influence the quality of the final reconstructed image.

2. Signal model

Consider an M-pixel image. Let $\boldsymbol{\rho} = [\rho_m]$ be a complex-valued M-vector (of which each entry is called a pixel) consisting of the MR contrasts. In general, the coil sensitivity also exhibits spatial variation; therefore let the $M \times C$ complex-valued matrix $\mathbf{C} = [c_{mk}]$ consist of coil sensitivity values, whose (m, k)th entry is the sensitivity of coil k at pixel m. In general, if $\mathbf{S} = [s_{mk}]$ denotes the measurement matrix, the measurement of coil k for pixel m is given by [4]

$$s_{mk} = \rho_m c_{mk}^* + \sigma n_{mk}, \tag{1}$$

where $*$ denotes the complex conjugate, $\mathbf{N} = [n_{mk}]$ is a complex-valued measurement noise matrix. The rows of N are usually assumed to be independent, but its columns are correlated with covariance matrix $\mathbf{Q} = [q_{mk}]$; i.e., letting δ_{mn} to be the Kronecker-delta function, $E[n_{mk}^{\mathrm{H}} n_{nl}] = q_{kl} \cdot \delta_{mn}$, where H stands for conjugate-transpose. This assumption is based on the rationale that the measurement noise at a certain pixel is usually correlated across different coils, but the noise is spatially white (if this is not the case, a spatial whitening procedure can be applied as pre-processing). It is also common to assume that the noise distribution is Gaussian [4]. In (1), we have introduced the parameter σ to allow manipulation of the noise-power without changing the shape of the noise probability distribution. In addition, we make the following assumptions on the second-order statistics of the complex-valued measurement noise: the real and imaginary parts:

(A1) are uncorrelated, and
(A2) have identical covariance.

These assumptions are common in signal processing due to the fact that complex noise usually emanates from a complex baseband representation of stationary narrowband noise. Combined with $E[n_{mk}^{\mathrm{H}} n_{nl}] = q_{kl} \cdot \delta_{mn}$, A1 and A2 lead to the following second-order noise statistics for pixel m:

$$\begin{aligned} E\left[\mathrm{Re}\left\{n_{mk}^{\mathrm{H}}\right\} \mathrm{Re}\left\{n_{ml}\right\}\right] &= E\left[\mathrm{Im}\left\{n_{mk}^{\mathrm{H}}\right\} \mathrm{Im}\left\{n_{ml}\right\}\right] \\ &= \frac{q_{kl}}{2}, \\ E\left[\mathrm{Re}\left\{n_{mk}^{\mathrm{H}}\right\} \mathrm{Im}\left\{n_{ml}\right\}\right] &= E\left[\mathrm{Im}\left\{n_{mk}^{\mathrm{H}}\right\} \mathrm{Re}\left\{n_{ml}\right\}\right] \\ &= 0. \end{aligned} \tag{2}$$

3. Maximum ratio combining

For complex-valued signals, assuming that the coil sensitivities are known, the SNR-optimal unbiased linear combination of the measurements for the mth pixel is given by the matched filter:

$$\hat{\rho}_m = \frac{\mathbf{s}_m^{\mathrm{T}}\mathbf{c}_m}{\mathbf{c}_m^{\mathrm{H}}\mathbf{c}_m}, \tag{3}$$

where T denotes transpose and the vectors $\mathbf{s}_m$ and $\mathbf{c}_m$ denote vectors of coil measurement and sensitivities for pixel m. The SNR-optimality of this reconstruction method among all linear combiners is proved by a straightforward application of the Cauchy–Schwartz inequality [6]. The corresponding SNR is $|\rho_m|^2\|\mathbf{c}_m\|^4/(\sigma^2\mathbf{c}_m^{\mathrm{H}}\mathbf{Q}\mathbf{c}_m)$, where $\|.\|_\alpha$ denotes the L_α vector norm, $\|.\|$ denotes the L_2 norm, and $|.|$ denotes complex magnitude. In practice, since the coil sensitivities are not known, this approach cannot be used.

4. Sum-of-squares reconstruction

A standard method for combining measurements acquired by multiple coils is sum-of-squares (SoS) reconstruction [10,11]. This commonly used method is a special case of a more general sum-of-powers approach, which we call the α-power reconstruction (Soα). With Soα-reconstruction, the pixel m is given by

$$\hat{\rho}_m = \left(\sum_{k=1}^{C} |s_{mk}|^\alpha\right)^{1/\alpha} = ||\mathbf{s}_m||_\alpha . \tag{4}$$

For SoS, α is equal to two. In the noise-free case, we have $\sigma = 0$, and the estimator in (4) yields the following pixel value:

$$\hat{\rho}_m(0) = |\rho_m| \cdot ||\mathbf{c}_m||_\alpha^{1/\alpha} . \tag{5}$$

For small noise power levels, the solution in (4) can be accurately approximated by a first-order Taylor expansion:

$$\hat{\rho}_m(\sigma) \approx \hat{\rho}_m(0) + \hat{\rho}'_m(0)\sigma. \tag{6}$$

This leads to the following asymptotic SNR expression:

$$\mathrm{SNR}_{\mathrm{So}\alpha} = \frac{E\left[|\hat{\rho}_m(0)|^2\right]}{\sigma^2 E\left[|\hat{\rho}'_m(0)|^2\right]}. \tag{7}$$

Assuming the pixel values and coil sensitivities are unknown deterministic quantities (treating them as uncorrelated random variables leads to similar results, where all deterministic quantities involving the image and coil sensitivities become expectations instead), from (5) we see that the numerator of (7) is given by

$$E\left[|\hat{\rho}_m(0)|^2\right] = |\hat{\rho}_m|^2\,\|\mathbf{c}_m\|_\alpha^2. \tag{8}$$

Under assumptions A1 and A2, the denominator of (7) can also be found to be (see Appendix A)

$$E\left[|\hat{\rho}'_m(0)|^2\right] = ||\mathbf{c}_m||_\alpha^{2(1-\alpha)} \cdot \mathbf{c}_m^{\mathrm{H}}\mathbf{Q}\mathbf{c}_m\mathbf{D}^{2(\alpha-2)}, \tag{9}$$

where $\mathbf{D} = \mathrm{diag}\{|c_{m1}|, \ldots, |c_{mC}|\}$. Substituting (8) and (9) in (7), we see that the asymptotic SNR of Soα is

$$\mathrm{SNR}_{\mathrm{So}\alpha} = \frac{|\boldsymbol{\rho}_m|^2\|\mathbf{c}_m\|_\alpha^{2\alpha}}{\sigma^2\mathbf{c}_m^{\mathrm{H}}\mathbf{Q}\mathbf{c}_m\mathbf{D}^{2(\alpha-2)}}. \tag{10}$$

Specifically for SoS ($\alpha = 2$), the SNR becomes

$$\mathrm{SNR}_{\mathrm{SoS}} = \frac{|\rho_m|^2\|\mathbf{c}_m\|_2^4}{\sigma^2\mathbf{c}_m^{\mathrm{H}}\mathbf{Q}\mathbf{c}_m}, \tag{11}$$

which indicates that if the coils and their physical configuration can be designed such that $\mathbf{c}_m^{\mathrm{H}}\mathbf{Q}\mathbf{c}_m$ is small, the SNR of the reconstructed image can be improved. This can be achieved by forcing the vector $\mathbf{c}_m$ to the subspace spanned by the eigenvectors of $\mathbf{Q}$ corresponding to the smaller eigenvalues. In practice, the noise covariance matrix $\mathbf{Q}$ typically is a function of the coil sensitivities or of the physical coil configuration. Therefore, the problem then becomes optimizing $\mathbf{c}_m^{\mathrm{H}}\mathbf{Q}(\mathbf{c}_m)\mathbf{c}_m$ by adjusting coil the configuration to achieve maximum SNR.

Consider the special case when $\mathbf{Q} = \mathbf{I}$. Then $\mathrm{SNR}_{\mathrm{SoS}} = |\rho_m|^2\|\mathbf{c}_m\|^2/\sigma^2$. This result was intuitively expected, because the SNR increases linearly with the signal power and it is inversely proportional to the noise power. Also we know from statistical estimation theory that if C independent measurements with identical variance are averaged, the estimation variance decreases by a factor of $\|\mathbf{c}_m\|^2$, which is the norm-square of the averaging weights—therefore the SNR increases by this factor.

5. Singular value decomposition reconstruction

The traditional SoS method works on a pixel-by-pixel basis, whereas in order to apply singular value decomposition (SVD) to acquire the image *vector*, an additional assumption is required. In SVD reconstruction it is assumed that the coil sensitivity remains approximately constant over a small region of the image. This is a reasonable assumption since the coil sensitivity profiles will vary smoothly over space, thus in a small neighborhood Ω, they can be assumed to be constant. Under this assumption, the signal model for the region Ω becomes

$$\mathbf{S} = \boldsymbol{\rho}\mathbf{c}^{\mathrm{H}} + \sigma\mathbf{N}, \tag{12}$$

where $\boldsymbol{\rho}$ is the $P \times 1$ image vector consisting of the pixel values in region Ω, $\mathbf{c}$ is the $C \times 1$ coil sensitivity vector consisting of the sensitivities of the coils in the phased array which are *constant* for all pixels in Ω. The overall $P \times C$ measurement matrix for Ω is $\mathbf{S}$, and the matrix of measurement noise is $\sigma\mathbf{N}$. We make the same assumptions on the noise statistics as before, i.e., $E[\mathbf{N}]=\mathbf{0}$ and $E[\mathbf{N}^{\mathrm{H}}\mathbf{N}]=\mathbf{Q}$. For the analysis of the SVD method, assumptions A1 and A2 are not required.

The SVD reconstruction simply follows from the fact that when the measurement is noise-free (i.e., $\sigma = 0$) the image vector for region Ω is the eigenvector of the matrix $\mathbf{S}\mathbf{S}^{\mathrm{H}} = \|\mathbf{c}\|^2\boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{H}}$ corresponding to the nonzero eigenvalue. In the general noisy case, the best image estimate is given by the minimizer of the least-squares criterion $\|\mathbf{S} - \boldsymbol{\rho}\mathbf{c}^{\mathrm{H}}\|^2$, which is equal to the eigenvector of $\mathbf{S}\mathbf{S}^{\mathrm{H}}$ corresponding to the largest eigenvalue. We have previously demonstrated that in real MRI data, the matrix $\mathbf{S}$ effectively has rank one in the regions where the signal has sufficient power [13]; hence the assumptions underlying the SVD method are valid. In the regions of the image dominated by noise, the assumption breaks down; however, the reconstructed image quality in these parts of the image is not of large interest anyway.

In the noisy case, the largest eigenvector of $\mathbf{S}\mathbf{S}^{\mathrm{H}}$ will be perturbed from its noise-free value of $\boldsymbol{\rho}$. When evaluating the SNR in the reconstructed image, we are interested in the power of the perturbation in this eigenvector. The problem of determining this perturbation power can be tackled more easily if we instead consider the matrix $\mathbf{S}^{\mathrm{H}}\mathbf{S}$. In particular,

$$\begin{aligned}\mathbf{S}^{\mathrm{H}}\mathbf{S} &= \left(\boldsymbol{\rho}\mathbf{c}^{\mathrm{H}} + \sigma\mathbf{N}\right)^{\mathrm{H}}\left(\boldsymbol{\rho}\mathbf{c}^{\mathrm{H}} + \sigma\mathbf{N}\right)\\ &= \|\boldsymbol{\rho}\|^2\mathbf{c}\mathbf{c}^{\mathrm{H}} + \sigma\mathbf{N}^{\mathrm{H}}\boldsymbol{\rho}\mathbf{c}^{\mathrm{H}} + \sigma\mathbf{c}\boldsymbol{\rho}^{\mathrm{H}}\mathbf{N} + \sigma^2\mathbf{N}^{\mathrm{H}}\mathbf{N}.\end{aligned} \tag{13}$$

For $\sigma = 0$, the nonzero eigenvalue of $\mathbf{S}^{\mathrm{H}}\mathbf{S}$ is observed to be $\lambda(0) = \|\boldsymbol{\rho}\|^2\|\mathbf{c}\|^2$. The first- and second-order derivatives of this eigenvalue evaluated at $\sigma = 0$ are (see Theorem 6.3.12, p. 372, of [6]):

$$\begin{aligned}&\lambda'(0) = 2\mathrm{Re}\{\mathbf{c}\boldsymbol{\rho}^{\mathrm{H}}\mathbf{N}\mathbf{c}\}/\|\mathbf{c}\|^2,\\ &\lambda''(0) = 2\mathbf{c}^{\mathrm{H}}\mathbf{N}^{\mathrm{H}}\mathbf{N}\mathbf{c}/\|\mathbf{c}\|^2.\end{aligned} \tag{14}$$

For small σ the nonzero eigenvalue of $\mathbf{S}^{\mathrm{H}}\mathbf{S}$ can be approximated by

$$\begin{aligned}\lambda(\sigma) &= \lambda(0) + \lambda'(0)\sigma + \tfrac{1}{2}\lambda''(0)\sigma^2\\ &= \|\boldsymbol{\rho}\|^2\|\mathbf{c}\|^2 + 2\mathrm{Re}\{\boldsymbol{\rho}^{\mathrm{H}}\mathbf{N}\mathbf{c}\}\sigma\\ &\quad + \mathbf{c}^{\mathrm{H}}\mathbf{N}^{\mathrm{H}}\mathbf{N}\mathbf{c}\sigma^2/\|c\|^2.\end{aligned} \tag{15}$$

Considering the image and coil sensitivity vectors to be unknown deterministic (once again taking them to be uncorrelated random vectors leads to similar results) and taking the expectation of (15) with respect to N results in

$$\begin{aligned}E[\lambda(\sigma)] &= \|\boldsymbol{\rho}\|^2\|\mathbf{c}\|^2 + 2\mathrm{Re}\{\boldsymbol{\rho}^{\mathrm{H}}E[\mathbf{N}]\mathbf{c}\}\sigma\\ &\quad + \mathbf{c}^{\mathrm{H}}E[\mathbf{N}^{\mathrm{H}}\mathbf{N}]\mathbf{c}\sigma^2/\|\mathbf{c}\|^2\\ &= \|\boldsymbol{\rho}\|^2\|\mathbf{c}\|^2 + \mathbf{c}^{\mathrm{H}}\mathbf{Q}\mathbf{c}\sigma^2/\|\mathbf{c}\|^2.\end{aligned} \tag{16}$$

We notice that the SNR in the reconstructed image is given by the two terms in this expression. Specifically, we can see that, for small noise, the SNR in the SVD reconstructed image will be

$$\mathrm{SNR}_{\mathrm{SVD}} = \frac{\|\boldsymbol{\rho}\|^2\|\mathbf{c}\|^4}{\sigma^2(\mathbf{c}^{\mathrm{H}}\mathbf{Q}\mathbf{c})}. \tag{17}$$

This SNR is identical to what was obtained for SoS. If the noise covariance matrix is $\mathbf{Q} = \mathbf{I}$, the SNR in (17) becomes $\|\boldsymbol{\rho}\|^2\|\mathbf{c}\|^2/\sigma^2$, which is P times larger than the result for SoS. This is so because the region Ω covers P pixels and $\mathrm{SNR}_{\mathrm{SVD}}$ considers the total signal power in Ω.

6. Normalized weighted coil average reconstruction

The applicability of the normalized coil average (NCA) method relies on the same assumptions as the SVD reconstruction technique. It is assumed that the coil sensitivities remain constant in small neighborhoods; therefore, the same signal model as in (12) holds. Specifically, for the kth coil the measurement vector is

$$\mathbf{s}_k = c_k^* \boldsymbol{\rho} + \sigma \mathbf{n}_k, \tag{18}$$

where $\mathbf{s}_k$ and $\mathbf{n}_k$ are the kth columns of $\mathbf{S}$ and $\mathbf{N}$ in (12). The normalized weighted coil average reconstruction is then [13]

$$\hat{\boldsymbol{\rho}} = \sum_{k=1}^{C} \alpha_k \frac{\mathbf{s}_k}{\|\mathbf{s}_k\|} = \boldsymbol{\alpha}^{\mathrm{T}} \mathbf{S} \mathbf{B}^{-1}, \tag{19}$$

where $\mathbf{B} = \mathrm{diag}\{\|\mathbf{s}_1\|, \ldots, \|\mathbf{s}_C\|\}$. Notice that in the noise-free case the estimate becomes

$$\hat{\boldsymbol{\rho}}(0) = \frac{\boldsymbol{\rho}}{\|\boldsymbol{\rho}\|} \mathbf{c}^{\mathrm{H}} \mathbf{D}^{-1} \boldsymbol{\alpha}. \tag{20}$$

If the linear combination weights are selected as $\boldsymbol{\alpha} = \mathbf{D}^{-1}\mathbf{c}/C$, the estimate will have unit norm (power).

The asymptotic SNR for this reconstruction method can be obtained using the expansion $\hat{\boldsymbol{\rho}}(\sigma) \approx \hat{\boldsymbol{\rho}}(0) + \hat{\boldsymbol{\rho}}'(0)\sigma$. Specifically, the approximate SNR is given by

$$\mathrm{SNR}_{\mathrm{NCA}} = \frac{E\left[\|\hat{\boldsymbol{\rho}}(0)\|^2\right]}{\sigma^2 E\left[\|\hat{\boldsymbol{\rho}}'(0)\|^2\right]}. \tag{21}$$

Under the same assumptions as SoS (including A1 and A2), the numerator is determined to be

$$E\left[\|\hat{\boldsymbol{\rho}}(0)\|^2\right] = \boldsymbol{\alpha}^{\mathrm{H}} \mathbf{D}^{-1} \mathbf{c} \mathbf{c}^{\mathrm{H}} \mathbf{D}^{-1} \boldsymbol{\alpha} \tag{22}$$

and the denominator (whose expression involves a potentially confusing form in vector–matrix notation) is, in scalar terms, calculated as (see Appendix B)

$$E\left[\|\hat{\boldsymbol{\rho}}'(0)\|^2\right] = \sum_{k=1}^{C} \sum_{l=1}^{C} \frac{q_{kl} \mathrm{Re}\{\alpha_k^* \alpha_l c_k c_l^*\} \mathrm{Re}\{c_k^* c_l\}}{\|\boldsymbol{\rho}\|^2 |c_k|^3 |c_l|^3}. \tag{23}$$

Substituting these results in (21) yields the SNR for NCA. In particular, for the usual special case that are considered in the previous sections, the SNR becomes

$$\mathrm{SNR}_{\mathrm{NCA}} = \frac{\|\boldsymbol{\rho}\|^2 \boldsymbol{\alpha}^{\mathrm{H}} \mathbf{D}^{-1} \mathbf{c} \mathbf{c}^{\mathrm{H}} \mathbf{D}^{-1} \boldsymbol{\alpha}}{\sigma^2 \boldsymbol{\alpha}^{\mathrm{H}} \mathbf{D}^{-1} \boldsymbol{\alpha}}. \tag{24}$$

We notice that, in (24), if the averaging weight vector is selected to be $\boldsymbol{\alpha} = \mathbf{D}\mathbf{c}$, then $\mathrm{SNR}_{\mathrm{NCA}} = \|\boldsymbol{\rho}\|^2 \|\mathbf{c}\|^2/\sigma^2$, which is identical to that of SoS and SVD for this situation. In practice, however, $\mathbf{c}$ is not known. In such cases, where there is no a priori knowledge about coil sensitivities, it is reasonable to use equal weights, i.e., $\alpha_k = 1/C$. For this weight assignment, defining $\mathbf{d} = \mathrm{diag}(\mathbf{D}^{-1}\} = [1/|c_1|, \ldots, 1/|c_C|]^{\mathrm{T}}$, the SNR becomes $\|\boldsymbol{\rho}\|^2 \mathbf{d}^{\mathrm{T}} \mathbf{c} \mathbf{c}^{\mathrm{H}} \mathbf{d}/(\mathbf{d}^{\mathrm{T}} \mathbf{d} \sigma^2)$. In general, this SNR can be expected to be smaller than that of SoS and SVD methods, since determining the optimal combination weights require the knowledge of the coil sensitivities.

7. Discussion

Recent trends in MRI technology indicate the need for fast and high-quality imaging techniques using multiple coils. In this correspondence, we have investigated the asymptotic SNR performance of three image combination methods for phased-array MRI. The analytical SNR expressions provided here facilitate an understanding for the design of optimal fast-imaging algorithms that produce high-quality images. In addition, our analytical expressions lead to interesting insights about the effect of system parameters, such as coil sensitivities and noise statistics. Future work may focus on optimizing phased-array configurations to maximize the image quality.

An interesting observation that emerged from our theoretical analysis of the different image combination methods is that asymptotically, the sum-of-squares method becomes SNR-optimal, since its performance converges to that of the maximum-ratio combining. SVD method, on the other hand, is an SNR-optimal method by definition, due to the fact that the SVD solution naturally appears as a consequence of minimum MSE estimation, and the SNR can be seen as the ratio of the signal power to the error MSE.

Appendix A.

The explicit expression for the sensitivity of the Soα solution to the noise standard deviation is found to be

$$\hat{\rho}'_m = \|\mathbf{s}_m\|_\alpha^{1-\alpha} \left(\sum_{k=1}^{C} |s_{mk}|^{\alpha-2} \mathrm{Re}\left\{ n_{mk}^* s_{mk} \right\} \right). \tag{A.1}$$

Evaluated at $\sigma = 0$, this becomes

$$\hat{\rho}'_m(0) = |\rho_m|^{-1}\|\mathbf{c}_m\|_\alpha^{1-\alpha} \times \left(\sum_{k=1}^{C} |c_{mk}|^{\alpha-2}\mathrm{Re}\left\{c^*_{mk}n^*_{mk}\rho_m\right\}\right). \quad \text{(A.2)}$$

Finally, we calculate

$$E\left[\left|\hat{\rho}'_m(0)\right|^2\right] = E\left[|\rho_m|^{-2}\,\|\mathbf{c}_m\|_\alpha^{2(1-\alpha)} \times \left(\sum_{k=1}^{C}\sum_{l=1}^{C} |c_{mk}|^{\alpha-2}\,|c_{ml}|^{\alpha-2} \times \mathrm{Re}\left\{c^*_{mk}n^*_{mk}\rho_m\right\}\cdot\mathrm{Re}\left\{c^*_{ml}n^*_{ml}\rho_m\right\}\right)\right]$$

$$= \frac{q_{kl}}{2}\|\mathbf{c}_m\|_\alpha^{2(1-\alpha)}\left(\sum_{k=1}^{C}\sum_{l=1}^{C}|c_{mk}|^{\alpha-2} \times |c_{ml}|^{\alpha-2}\,\mathrm{Re}\left\{c^*_{mk}c_{ml}\right\}\right), \quad \text{(A.3)}$$

where we have utilized

$$E\left[\mathrm{Re}\left\{c^*_{mk}n^*_{mk}\rho_m\right\}\cdot\mathrm{Re}\left\{c^*_{ml}n^*_{ml}\rho_m\right\}\right] = \frac{q_{kl}}{2}\,|\rho_m|^2\,\mathrm{Re}\left\{c^*_{mk}c_{ml}\right\} \quad \text{(A.4)}$$

which is derived (Appendix C) using A1 and A2. Finally, when everything is collected into vector–matrix form, (9) is obtained.

Appendix B.

In NCA, the sensitivity of the reconstructed image with respect to the noise power parameter σ can be found as

$$\hat{\boldsymbol{\rho}}'(\sigma) = \sum_{k=1}^{C}\alpha_k\left(\frac{\mathbf{n_k}}{\|\mathbf{s}_k\|} - \frac{\mathbf{s}_k\left(\mathbf{n}_k^{\mathrm{H}}\mathbf{s}_k + \mathbf{s}_k^{\mathrm{H}}\mathbf{n}_k\right)}{2\|\mathbf{s}_k\|^3}\right). \quad \text{(B.1)}$$

Evaluated at $\sigma = 0$ this becomes

$$\hat{\boldsymbol{\rho}}'(0) = \sum_{k=1}^{C}\alpha_k \times \left(\frac{\mathbf{n}_k}{\|\boldsymbol{\rho}\||c_k|} - \frac{c_k^*\boldsymbol{\rho}\left(c_k^*\mathbf{n}_k^{\mathrm{H}}\boldsymbol{\rho} + c_k\boldsymbol{\rho}^{\mathrm{H}}\mathbf{n}_k\right)}{2\|\boldsymbol{\rho}\|^3|c_k|^3}\right). \quad \text{(B.2)}$$

The norm of this vector, after some algebraic simplifications, is obtained as

$$\|\hat{\boldsymbol{\rho}}'(0)\|^2 = \sum_{k=1}^{C}\sum_{l=1}^{C}\frac{q_{kl}\mathrm{Re}\{\alpha_k^*\alpha_l\}}{\|\boldsymbol{\rho}\|^2|c_k|\,|c_l|} - \frac{\mathrm{Re}\{\alpha_k^*\alpha_l c_k\boldsymbol{\rho}^{\mathrm{H}}\mathbf{n}_l\}\mathrm{Re}\{c_k^*\mathbf{n}_k^{\mathrm{H}}\boldsymbol{\rho}\}}{\|\boldsymbol{\rho}\|^4|c_k|^3|c_l|} - \frac{\mathrm{Re}\{\alpha_k^*\alpha_l c_l^*\mathbf{n}_k^{\mathrm{H}}\boldsymbol{\rho}\}\mathrm{Re}\{c_l^*\mathbf{n}_l^{\mathrm{H}}\boldsymbol{\rho}\}}{\|\boldsymbol{\rho}\|^4|c_k|\,|c_l|^3} + \frac{\mathrm{Re}\{\alpha_k^*\alpha_l c_k c_l^*\}\mathrm{Re}\{c_k^*\mathbf{n}_k^{\mathrm{H}}\boldsymbol{\rho}\}\mathrm{Re}\{c_l^*\mathbf{n}_l^{\mathrm{H}}\boldsymbol{\rho}\}}{\|\boldsymbol{\rho}\|^4|c_k|^3|c_l|^3}. \quad \text{(B.3)}$$

In order to find the expected value of (B.3), we resort to the derivation in Appendix C once again. This results in the following final explicit expression for the expected value of (B.3):

$$E\left[\|\hat{\boldsymbol{\rho}}'(0)\|^2\right] = \sum_{k=1}^{C}\sum_{l=1}^{C}\frac{q_{kl}\mathrm{Re}\{\alpha_k^*\alpha_l c_k c_l^*\}\mathrm{Re}\{c_k^* c_l\}}{\|\boldsymbol{\rho}\|^2|c_k|^3\,|c_l|^3}. \quad \text{(B.4)}$$

Appendix C.

A key tool that was utilized in the preceding appendices is presented here. Specifically, we are interested in evaluating the following expression:

$$E\left[\mathrm{Re}\{c_1^*\mathbf{n}_1^{\mathrm{H}}\boldsymbol{\rho}\}\mathrm{Re}\{c_2^*\mathbf{n}_2^{\mathrm{H}}\boldsymbol{\rho}\}\right] = E\left[c_{1r}c_{2r}(\mathbf{n}_{1r}^{\mathrm{T}}\boldsymbol{\rho}_r + \mathbf{n}_{1i}^{\mathrm{T}}\boldsymbol{\rho}_i)(\mathbf{n}_{2r}^{\mathrm{T}}\boldsymbol{\rho}_r + \mathbf{n}_{2i}^{\mathrm{T}}\boldsymbol{\rho}_i) + c_{1r}c_{2i}(\mathbf{n}_{1r}^{\mathrm{T}}\boldsymbol{\rho}_r + \mathbf{n}_{1i}^{\mathrm{T}}\boldsymbol{\rho}_i)(\mathbf{n}_{2r}^{\mathrm{T}}\boldsymbol{\rho}_i - \mathbf{n}_{2i}^{\mathrm{T}}\boldsymbol{\rho}_r) + c_{1i}c_{2r}(\mathbf{n}_{1r}^{\mathrm{T}}\boldsymbol{\rho}_i - \mathbf{n}_{1i}^{\mathrm{T}}\boldsymbol{\rho}_r)(\mathbf{n}_{2r}^{\mathrm{T}}\boldsymbol{\rho}_r + \mathbf{n}_{2i}^{\mathbf{T}}\boldsymbol{\rho}_i) + c_{1i}c_{2i}(\mathbf{n}_{1r}^{\mathrm{T}}\boldsymbol{\rho}_i - \mathbf{n}_{1i}^{\mathrm{T}}\boldsymbol{\rho}_r)(\mathbf{n}_{2r}^{\mathrm{T}}\boldsymbol{\rho}_i - \mathbf{n}_{2i}^{\mathrm{T}}\boldsymbol{\rho}_r)\right]. \quad \text{(C.1)}$$

In (C.1), c is a complex scalar, $\mathbf{n}_1$, $\mathbf{n}_2$, and $\boldsymbol{\rho}$ are complex vectors. In addition, the subscripts r and i denote real and imaginary parts in the expansion. Under assumptions A1 and A2, the resulting statistical properties for the real and imaginary parts of the noise lead to the following simplification for (C.1), where

$$q_{12} = E[\mathbf{n}_1^{\mathrm{H}}\mathbf{n}_2]:$$
$$E[.] = \frac{q_{12}}{2}(c_{1r}c_{2r} + c_{1i}c_{2i})(\|\boldsymbol{\rho}_r\|^2 + \|\boldsymbol{\rho}_i\|^2). \qquad \text{(C.2)}$$

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