Geometric Structure of Sum-of-rank-1 Decompositions for n-Dimensional Order-p Symmetric Tensors

Olexiy Kyrgyzov
Department of Computer Science & Electrical Engineering
Oregon Health and Science University
Portland, USA
kyrgyzov@csee.ogi.edu

Deniz Erdogmus
Department of Computer Science & Electrical Engineering
Oregon Health and Science University
Portland, USA
derdogmus@ieee.org

Abstract — **The canonical sum-of-rank-one decomposition of tensors is a fundamental linear algebraic problem encountered in signal processing, machine learning, and other scientific fields. Current algorithms that emerge from CANDECOMP or PARAFAC formalisms rely on the basic definition of tensor decomposition that describes rank as the minimum number of vectors that are needed to reconstruct the tensor using outer product linear combinations, which is an extension of the same property of matrix rank. In this paper, we reinterpret the orthogonality condition of symmetric matrix eigenvectors as a geometric constraint on the coordinate frame formed by the eigenvectors and relaxing the orthogonality, we develop a set of structured-bases that can be utilized to decompose any symmetric tensor into its sum-of-rank-one (canonical) decomposition. The eigenvectors of order-p tensors are observed to form a frame where the angle between various pairs of eigenvectors are integer multiples of π/p. Validation of the proposed geometric structure and demonstration of decomposition accuracies obtained using these frames (at the level of a computer's numerical-ε) are provided.**

I. Introduction

In this paper, we propose a geometrical constraint for the eigenvectors of an n-dimensional order-p symmetric tensor. The particular cases of tensor decompositions corresponding to the well known n-dimensional order-2 tensors (matrices) and our recent solution for the 2-dimensional order-p tensors will be presented first to provide motivation and intuition. The proposition generalizes the geometrical orthogonal coordinate frame interpretation of symmetric real-valued matrix eigendecompositions to any-order symmetric real-valued tensors. This paper is important for the community to disseminate our proposal at an early stage in order to attract the interest and effort of other researchers working in the tensor decomposition field to developing the theoretical and practical aspects of the proposed tensor decomposition approach.

Since tensors and their eigendecompositions are not yet widely known, we include a basic introduction to the relevant concepts, as well as an explanation of the traditional CANDECOMP/PARAFAC (CP) approaches that attempted to solve this problem in recent decades. Our goal is to achieve a reasonably self-contained text to attract the attention of readers who might be unfamiliar with the topic. Consequently, although our main interest is in symmetric real tensors, we will start by discussing the generalization of singular value decompositions of nonsymmetric matrices to arbitrary real-valued tensors. The specific case of symmetric tensors is relatively simpler because one needs not deal with eigenvectors from left, right, and other index directions (modes), since all of these vectors corresponding to a singular value are identical for symmetric tensors (as the left/right eigenvectors of symmetric matrices are identical).

We first describe tensors, their CP and Tucker decompositions, which are utilized to approximate a given tensor with a lower rank tensor. Existing approaches tackle the problem of tensor decompositions from a low-rank approximation perspective in the sense that when tensor is approximated to some negligible error-norm by adding some small number of terms (e.g., rank-1 tensors in CANDECOMP) the algorithm is assumed to converge and the resulting decomposition is happily used for data compression or denoising purposes. It is not difficult to see that these existing approaches, although emerging from the singular value decomposition and deflation practices, cannot be rigorously viewed as eigenvector decompositions, since (i) the starting point of such algorithms is not a rigorous definition of eigenvectors for an order-p tensor, which due to the isomorphism with order-p polynomial spaces would not have eigenvectors in the usual bilinear-algebraic sense, (ii) they yield decreasing reconstruction error-norms as the number of rank-1 terms is increased, however our experience showed that for randomly selected tensors, typically the algorithms keep on going to generating infinite terms if not stopped artificially (e.g., by checking reconstruction error-norm versus some threshold), and (iii) disturbingly, when one compares the best rank-1 approximation to the best rank-2 approximation of a tensor, the rank-1 solution does not appear in the latter as a component, thus desirable deflation rules do not apply.

Our goal, in this paper, is to motivate that the eigenvectors (more appropriately, invariant vectors) of a tensor should have a geometric constraint that enables them to form a frame of vectors, which then could be jointly rotated in the corresponding vector space in order to identify the correct orientation of the eigenvector frame and the corresponding eigenvalues. Tensor analysis emerges in many fields of applications such as signal [1] and image [2] processing, factor analysis [3], speech, and telecommunications. As tensors are related to higher-order statistics through joint moments and cumulants of vector-valued random processes, we believe, they will play an increasingly more important role in statistical signal processing. A useful tensor eigendecomposition definition accompanied by computationally efficient tensor eigendecomposition algorithms will be key to widespread utilization of these multilinear objects.

II. Tensors And Low-Rank Approximations

The term tensor has different definitions in physics and mathematics. In mathematics it represents a nonlinear (e.g., polynomial) function or a multiway data array. In physics tensor fields that describe the properties of a system in a geometric space are referred to as tensors in short. In computer vision papers, it is possible to see traces of both nomenclatures, adding to the confusion. In signal processing, we refer to multiway arrays, thus tensors are generalization of scalars (order-0), vectors (order-1), and matrices (order-2) to arrays with more entry indices. In the literature, there is also some confusion about the use of the term tensor rank. While some would refer to vectors as rank-1 and to matrices as rank-2 tensors, we use the term order-*p* to denote the number of indices

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required to identify an element (and also due to the link with order-p polynomials). Furthermore, in signal processing papers, typically, the term tensor-rank is utilized to indicate the number of rank-1 tensors (outer product of p vectors for nonsymmetric tensors or the p-times outer product of a single vector with itself for symmetric tensors) one needs to add to obtain the original tensor, similar to matrix rank as defined by the number of eigenvectors with nonzero eigenvalues. There are also multiple definitions of tensor rank; we are interested in the version that reduces to matrix rank in the usual sense.

An order-p n-dimensional hypercube tensor is a mathematical object that has n^p elements and each of them can be reached by a unique p-element index vector. We can represent a tensor as a multidimensional and multiorder array of data that obeys certain transformation rules and operations on arrays such as addition, multiplication, permutation of indices, and elementwise operations. Inner product spaces can also be defined on tensor objects. The most widely recognized approaches for low-rank tensor decompositions can be presented by two models: (1) canonical decomposition (CANDECOMP) by Carrol and Chang [4] or alternatively parallel factor analysis (PARAFAC) by Harshman [3]; (2) the Tucker [5] model of decomposition. For CP model tensor can be represented as a sum of rank-1 tensors. The number of sum terms depends on error of decomposition and generally we can say that series of sums generated with this definition can have infinite terms. The advantage of this definition is that it leads to a list of linear combination coefficients that are interpreted like singular values. As opposed to the CP, Tucker's proposed decomposition factors tensors as a finite sum, but not necessarily composed of rank-1 terms. This leads to the introduction of a nondiagonal core tensor. We use the following notational conventions: indices are denoted by $i_1,\ldots,i_p$ or similarly using other lowercase letters that will be clear from the context. We denote vectors by lowercase boldface letters, matrices and other higher order tensors with uppercase boldface letters. Various conventions are found in the literature since the community did not arrive at a consensus yet [6,7]. Let **A** be an order-p tensor of dimensions $n_1\times n_2\times\cdots\times n_p$. The k^{th} mode (or way) of **A** is of dimension n_k. For example a tensor of size 2×2×2 is 2-dimensional order-3 tensor. To avoid confusions, it will be assumed that $n_k>1$ for each mode of a tensor (i.e., matrices are order-2 tensors, not order-3 with a singleton mode). We denote the index of a single element of a tensor by subscripts: $\mathbf{A}_{i_1\ldots i_p}$.

The CP representation of tensors in decomposed lower rank form uses the sum of rank-1 tensors. A rank-1 tensor is one that can be written as the p-way outer product of vectors:

$$\mathbf{T} = \lambda\cdot\mathbf{u}^{(1)}\circ\mathbf{u}^{(2)}\circ\cdots\circ\mathbf{u}^{(p)} \tag{1}$$

where λ is a scalar multiplicative factor and each $\mathbf{u}^{(k)}$ is an n_k-dimensional vector. The $\circ$ symbol denotes outer product, so:

$$\mathbf{T}_{i_1\ldots i_p} = \lambda\cdot\mathbf{u}^{(1)}_{i_1}\circ\mathbf{u}^{(2)}_{i_2}\circ\cdots\circ\mathbf{u}^{(p)}_{i_p} \tag{2}$$

where $\mathbf{u}_i$ denotes the i^{th} entry of vector **u**. The best rank-1 approximation problem for matrices is also the same as the Alternating Least Squares algorithm for fitting a rank-1 CP model. This approximation problem is that, given a tensor **A**, we want to find a tensor **T** as in (1) such that ||**A-T**|| is as small as possible, according to a suitable tensor norm (usually Frobenius, but other measures are employed).[1] The higher-order power method computes a **T** that approximately solves this problem. This latter method works as follows: fix all **u**-vectors except $\mathbf{u}^{(1)}$ and optimize for $\mathbf{u}^{(1)}$ with a fixed-point iteration; after convergence, repeat the procedure for $\mathbf{u}^{(2)}$, $\mathbf{u}^{(3)}$, …, cycling through the indices until the specified number of iterations is exhausted (or some other stopping criterion is met). Details of this algorithm can be found in [8]. When the stopping criterion is met, the algorithm proposes a CP-type approximation to the original tensor **A** in a form that is the multilinear generalization of the best low-rank approximation problem for matrices:

$$\mathbf{A} = \sum_{l=1}^{r}\lambda_l\cdot\mathbf{u}_l^{(1)}\circ\mathbf{u}_l^{(2)}\circ\cdots\circ\mathbf{u}_l^{(p)} \tag{3}$$

Here $\mathbf{u}_l^{(k)}$ is the l^{th} singular vector along the k^{th} mode, and λ_l is a linear combination coefficient (also called singular value, but mistakenly in our belief, since low-rank approximations do not obey deflation invariance and these so-called singular values are not invariant in any sense). If exact equality is achieved as in (3) with the minimum possible integer r, then $rank_{\otimes}\mathbf{A}=r$. The Tucker decomposition [5], alternatively referred to as a rank-(r_1,r_2, …, r_p) decomposition [8], is the multilinear-rank tensor that is given by

$$\mathbf{A} = \sum_{l_1=1}^{r_1}\cdots\sum_{l_p=1}^{r_p}\mathbf{\Lambda}_{l_1\ldots l_p}\mathbf{u}^{(1)}_{l_1}\circ\cdots\circ\mathbf{u}^{(p)}_{l_p} \tag{4}$$

Note that the sum of Tucker ranks trivially provides an upper bound for the tensor-rank of the type shown in (3). Here $\mathbf{\Lambda}$ is a tensor of dimensions $r_1\times r_2\times\cdots\times r_p$. This tensor is often called the *core array* or the *core tensor* [9]. The higher-order orthogonal iteration algorithm [8], which is the multilinear generalization of the best rank-**r** approximation problem for matrices, finds the best rank-(r_1,r_2,…,r_p) approximation of a tensor. Also note that the CP decomposition is a special case of the Tucker decomposition where $r=r_1=r_2=\cdots=r_p$ and $\mathbf{\Lambda}$ is a diagonal tensor. Also, the Tucker decomposition can be regarded as an *inefficient* CP decomposition.

III. Eigenvectors Of A Symmetric Tensor

We deal with the problem of real-symmetric tensors because the decomposition of such a tensor into a sum of rank-1 tensors utilizes basis tensors that are p-way outerproducts of the same vector. That is, if A is symmetric (the value of the entries at any permutation of a given index vector are identical, similar to symmetric matrices having the same value at permuted row-column index locations), the rank-1 decomposition of this tensor is of the form

$$\mathbf{A} = \sum_{l=1}^{r}\lambda_l\mathbf{u}_l^{\circ p} \tag{5}$$

where $\mathbf{u}^{\circ p}$ denotes the p-way outer-product of the vector u (e.g., $u^{\circ 2}=uu^T$). Similar to nonsymmetric matrices, arbitrary tensors can be subjected to various inner products along different modes in order to obtain symmetric tensors whose eigenvectors relate to the mode-specific singular-vectors that one would seek for the rank-1 decompositions (e.g., for a matrix **A**, the left and right singular vectors could be obtained as eigenvectors of $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$). Therefore, we believe as a first step, it is sufficient to focus on symmetric tensors. The latter also arises most commonly in higher-order statistical signal processing problems in the form of moment or cumulant tensors thus are of specific interest themselves.

Next we present solutions for three type of tensors; two of them are marginal cases of dimension and order, and the third is the general case of n-dimensional order-p tensor decompositions, which reduces to the former two as particular cases.

A. Order-2 n-Dimensional Tensor

Here we briefly describe well known theory about the eigendecomposition of symmetric matrices by Jacobi algorithm. A symmetric n-dimensional order-2 tensor is a symmetric matrix. Any eigenvector basis for a real symmetric matrix is orthogonal, and can

[1] It is important to note that, as demonstrated by de Silva and Lim [10], the space of tensors might not have a norm that allows traditional signal processing understanding of "zero-error-norm iff identical objects" which is valid in Hausdorff spaces. Nevertheless, whether this mathematical inconvenience is a practical problem or not is yet to be discussed.

always be made into an orthonormal basis. Thus a real n-dimensional symmetric matrix can be decomposed as

$$\mathbf{A} = \sum_{l=1}^{n} \lambda_l \mathbf{u}_l^{\circ 2} \qquad (6)$$

where $\mathbf{U} = \{\mathbf{u}_l, l=1,...,n\}$ is orthogonal n-dimensional basis.

For numerical determination of (6) we can use, for instance, the Jacobi algorithm [11] that tries to find $m_r(n)=\binom{n}{2}$ rotation angles $\{\theta_{ij}, i=1,...,n-1, j=i+1,...,n\}$, such that we can construct a rotation matrix $R(\theta_{ij})$ in plane (i,j) with angle θ_{ij}. A set of rotation matrices completely describes the orientation of the orthonormal basis in the n-dimensional space or, in other words, the eigenvectors in the decomposition of (6). This eigendecomposition solution consists of $m_r(n)$ rotation angles and n eigenvalues. The number of unique elements of a symmetric n-dimensional matrix $\mathbf{A}$, $n(n+1)/2$, equals the sum $n+m_r(n)$. Solving for the rotation matrices $R(\theta_{ij})$, we can get the orthonormal system of eigenvectors $\mathbf{U}=\{\mathbf{u}_l, l=1,\ldots,n\}$:

$$\mathbf{U} = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \mathbf{R}(\theta_{ij}), \quad \mathbf{U}\mathbf{U}^T = \mathbf{I} \qquad (7)$$

where $\mathbf{u}_l$ is the l^{th} column of matrix $\mathbf{U}$. Due to orhtonormality of $\mathbf{U}$, the eigenvalues are uniquely identified by the Frobenious inner-product vector between the target matrix and the basis matrices:

$$\begin{aligned} <\mathbf{u}_l^{\circ 2}, \mathbf{A}>_F &= \sum_{i=1}^{n} \lambda_i <\mathbf{u}_l^{\circ 2}, \mathbf{u}_i^{\circ 2}>_F \\ &= \sum_{i=1}^{n} \lambda_i (\mathbf{u}_l^T \mathbf{u}_i)^2 = \lambda_l \end{aligned} \qquad (8)$$

B. Order-p 2-dimensional Tensor

In this section we focus on the 2-dimensional case where unit-length eigenvectors can be parameterized by one rotation angle. Let A be a 2-dimensional order-p real-symmetric tensor. The number of unique elements (not repeated due to symmetry) for this tensor can be shown to be p+1. Our goal is to find a decomposition of the type shown in (5) uniquely under certain acceptable permutations of the solution, such as the index shuffling of the eigenvectors and eigenvalues. For such uniqueness to be possible, the number of unique elements (equations) on the left-hand side of (5) must be equal to the number of parameters (unknowns) on the right-hand side. This implies that the number of eigenvalues, r, plus the number of parameters that characterize r corresponding vectors, s, (at least 1 angle if all vectors are fixed relative to each other and at most r angles if all vectors are free to take arbitrary orientations, as assumed in CP) is equal to p+1: that is (r+s)=(p+1).

It is desirable that the parameterization proposed for the 2-dimensional symmetric order-p tensor holds for order-0, order-1, and order-2 tensors. For scalars (p=0), r=1 and s=0, since there are no vectors to parameterize. For vectors (p=1), r=1 (± the norm of the vector) and s=1 (the orientation angle of the vector in 2-dimensional space). For symmetric matrices (p=2), r=2 and s=1 (since both vectors are defined by one rotation angle and the vectors are always orthogonal to each other). Generalizing this pattern using induction, we require that r=p; that is, the number of eigenvalues for an order-p 2-dimensional symmetric tensor is also p. Consequently, this leaves us only one parameter to characterize all corresponding eigenvectors: s=1. The latter is only possible if the eigenvectors form a fixed frame (relative pairwise angles are fixed a priori) and the whole frame is rotated at once using a single rotation angle (the free parameter) in order to achieve equality in (5). At the correct frame orientation, the eigenvalues obtained using the procedure in (8) will yield an exact decomposition of the tensor; at arbitrary frame orientations, these estimates of eigenvalues will not yield equality in (5).

An additional property that we desire from the eigenvector frame is "rotation-invariance". Considering again the 2-dimensional case, in particular, for vectors, if the basis vector is rotated by π radians, then one can change the sign of the scale factor (eigenvalue, or the norm parameter) to obtain the original vector; for matrices, if the orthogonal eigenvector frame is rotated by multiples of π/2 radians, the eigenvectors switch their indices and/or signs, thus simply switching the indices and signs of the corresponding eigenvalues results in the same matrix. In general, we want the eigenvectors of the order-p tensor to be separated by exactly π/p radians so that if the frame consisting of p vectors is rotated at angles that are integer multiples of this angle, the eigenvectors switch indices and/or signs, hence similar modification on the corresponding eigenvalues preserve the equality in (5).

Incorporating these conditions into the design of the rank-1 sum decomposition on the right-hand side of (5), we obtain that real-symmetric, 2-dimensional order-p tensor A has the following decomposition in the CP-sense, but utilizing a specific geometrical structure for its decomposition as opposed to the arbitrary vectors generated by some algorithms:[2]

$$\mathbf{A} = \sum_{l=1}^{p} \lambda_l(\theta) \cdot \mathbf{u}_l^{\circ p}(\theta), \ \mathbf{u}_l = \begin{bmatrix} \cos(\theta + (l-1)\pi/p) \\ \sin(\theta + (l-1)\pi/p) \end{bmatrix} \qquad (9)$$

In this case, a simple line search for θ in the interval [0,π/p) is sufficient (since the reconstruction error norm will be periodic by this amount). For each value of this frame orientation angle, the eigenvalues can be obtained, for instance, via least-Frobenious-norm-squared fitting (essentially least squares) in an analytical manner – the equations are omitted here for brevity. Employing the Gram-Schmidt orthogonalization, the eigenvalue vector $\boldsymbol{\lambda}$ is uniquely identified by the inner-product matrix between the basis tensor pairs and the inner-product vector between the target tensor and the basis tensors as $\boldsymbol{\lambda}(\boldsymbol{\theta}) = \mathbf{B}^{-1}\mathbf{c}(\boldsymbol{\theta})$. Here the matrix $\mathbf{B}$ (invariant with respect to $\boldsymbol{\theta}$, since the pairwise angles between the eigenvectors are fixed by the frame) and the vector $\mathbf{c}$ are defined elementwise as follows, assuming the Frobenius tensor inner product as in (8):

$$\mathbf{B}_{ij} = <\mathbf{u}_i(\boldsymbol{\theta}), \mathbf{u}_j(\boldsymbol{\theta})>^p, \quad \mathbf{c}_i(\boldsymbol{\theta}) = <\mathbf{u}_i^{\circ p}(\boldsymbol{\theta}), \mathbf{A}>_F \qquad (10)$$

where $i,j=1,\ldots,p$. Specifically note that each entry of $\mathbf{B}$ reduces to the following: $B_{ij}=\cos^p((i-j)\pi/p)$. For symmetric matrices, this matrix is simply identity.

C. Order-p n-Dimensional Tensor

In this section we present eigendecomposition of n-dimensional order-p symmetric tensors that includes previous sections as particular cases. The number of unique elements of a symmetric n-dimensional order-p tensor relies on the triangular numbers: $m_u(n,p)=\binom{n+p-1}{p}$. Based on the two special cases examined above we conclude that the decomposition of any symmetric tensor must consist of some fixed frame of vectors rotated in n-dimensional space and any angle between pairwise vectors must be constant and depends on order p. As in matrices, we need $m_r(n)$ rotation angles to characterize the orientation of the eigenvector frame. So we can decompose any symmetric n-dimensional order-p tensor as a finite sum of rank-1 tensors as in (5), where the number of vectors is: $r=\binom{n+p-1}{p}-\binom{n}{2}$.

[2] Note that a symmetric 2-dim matrix can be expressed as the sum of two rank-1 matrices constructed using any set of two linearly independent vectors rotated properly, thus a CP decomposition with different error norm measures would yield different such decompositions. The situation is the same with tensors.

To obtain the decomposition numerically, we construct a frame of r initial vectors $\mathbf{U}$ and optimize the rotation angles $\boldsymbol{\theta}$ such that the Frobenious error norm is minimized. For a given candidate frame orientation, the eigenvalues are always obtained using (10). The frame consists of vectors that are rotations of, for instance, the first column of the n-dimensional identity matrix multiplied by a rotation matrix with all possible permutations of angles that are multiples of π/p in consecutive dimension index pairs. Specifically, this leads to a system of $n^{(p-1)}$ vectors:

$$\mathbf{U}_{initial} = \left\{ \left(\prod_{i=1}^{n-1} \mathbf{R}_{i,i+1}(k_i \pi / p) \right) [1 \quad 0 \quad \cdots \quad 0]^T \right\} \qquad (11)$$

where the rotation matrices are always multiplying from the left. We denote the matrix that is used to obtain each vector in the frame as $\mathbf{R}_{k_{n-1}\ldots k_1}$. Here $\mathbf{R}_{i,i+1}(k_i \pi / p)$ is a rotation matrix in the plane $(i,i+1)$ with angle $k_i\pi/p$ and each index $k_i \in \{0,\ldots,(p-1)\}$.

Note that the number of vectors in this frame is more than the number of vectors needed, r. Any subset that yields a full-rank $\mathbf{B}$ in (10) can be selected and utilized. The elimination procedure in Fig. 1 yields the exact number of vectors prescribed and yields a full-rank $\mathbf{B}$. After elimination and searching for the optimal frame orientation, the eigenvalues are obtained using (10) and the final eigenvectors are expressed as (all rotations multiply from left):

$$\mathbf{U} = \left(\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \mathbf{R}(\theta_{ij}) \right) \mathbf{U}_{initial} \qquad (14)$$

IV. Numerical Demonstrations

A visual illustration of the proposed eigenvector frame scheme for real-symmetric 2-dimensional order-p tensors and the frame of 7 eigenvectors after the decomposition of a random 3-dimensional order-3 tensor is provided in Fig. 2 for illustration. The number of parameters to be determined (frame orientation angles and eigenvalues) is equal to the number of unique elements in the tensor. The reconstruction error Frobenius tensor-norm between the two sides of (6) are shown as a function of the frame angle in Fig. 3 (left) for 2-dimensional cases. As expected, for order 2, 4, and 7 tensors, the error completes 2, 4, and 7 cycles respectively. In Fig. 3, we also present the average normalized Frobenious error norm-squared per tensor entry for randomly generated tensors of orders 2 to 5 for dimensions 2 to 5.[3] The optimal orientation is identified by a search procedure similar to stochastic annealing. The reconstruction error-squared value averages around Matlab's minimum of 10^{-32} (e.g., check Matlab's internal eigendecomposition routine *eig*) for small order small-dimensional tensors. Errors of decomposition grows exponential due to the number of elements in the high-order high-dimensional tensors growing combinatorially combined with fast numerical degradation due to matrix ill-conditioning.

V. Conclusion

In previous work on CP and Tucker decompositions of tensors, researchers have proposed numerical algorithms for determining low-rank approximations to higher-order tensors in a manner similar to the low-rank representation capabilities of matrix eigenvectors. We propose a new direction to this problem by introducing geometric constraints to the eigenvector set for the general case of n-dimensional order-p symmetric tensors. The ideas presented here will be generalized to nonsymmetric tensors in future publications. Numerically more stable techniques will also have to be developed.

[3] Specifically, for a random tensor $\mathbf{A}$ and its reconstruction $\mathbf{T}$ with optimally selected $\boldsymbol{\theta}$, $E[\ \|\mathbf{A}-\mathbf{T}\|_F^2 \ /\ \|\mathbf{A}\|_F^2]/n^p$ is calculated where expectation is the average across 1000 random symmetric tensors.

$Initialize : Indices = \{1\}$

$for\ i = 2 : n^{(p-1)}$

$\quad TInd = Indices \cup \{i\},\ \mathbf{U} = \mathbf{U}_{initial}(TInd),\ [\mathbf{A}_{kl}] = [(\mathbf{u}_k^T \mathbf{u}_l)^p]\ \ k,l \in TInd$

$\quad if rank(\mathbf{A}) = |TInd|,\ Indices = TInd,\ end$

end

$\mathbf{U}_{initial} = \mathbf{U}_{initial}(TInd)$

Figure 1. A procedure for getting r vectors with a full-rank $\mathbf{B}$.

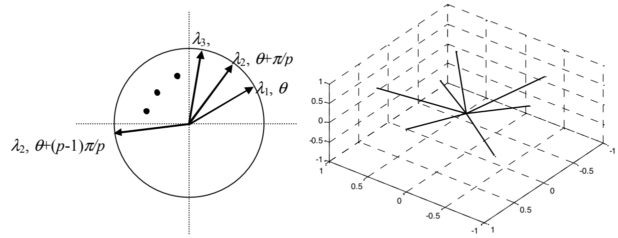

Figure 2. Illustration of the proposed eigenvector frame in terms of the angles and eigenvalues associated with each vector. (Left) for order-p 2-dimensional tensors and (Right) for order-3 3-dimensional tensors.

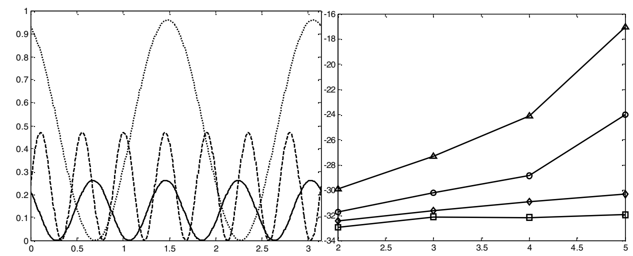

Figure 3. (Left) Frobenious norm squared of the reconstruction error for tensors of order 2 (dot), 4 (solid), and 7 (dash) versus $\theta \in [0,\pi)$. (Right) Average normalized reconstruction error-squared level for the proposed eigendecomposition technique for random tensors of orders 2 to 5 for dimensions 2 (square), 3 (diamond), 4 (circle), 5 (triangle).

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